Tutorial 6 2022.11.2

6.1 Midterm question 8

Problem 6.1

1. Let f and g be continuous on the region D in $\mathbb{R}^3$. Prove the inequality
$$2\iiint_D |fg|dV \leq \alpha^2 \iiint_D f^2 dV + \frac{1}{\alpha^2}\iiint_D g^2 dV,$$
where α is a positive number. Hint: Use $(a \pm b)^2 \geq 0$.
2. Prove
$$\iiint_D |fg|dV \leq \sqrt{\iiint_D f^2 dV}\sqrt{\iiint_D g^2 dV}$$
Hint: Make a good choice of α in the first inequality of 1..

Proof Assume 1. let's prove 2.. If $\iiint_D f^2 dV = 0$, then $f = 0$ on D, so $\iiint_D |fg|dV = 0$ and the inequality holds.

Therefore we may assume $\iiint_D f^2 dV \neq 0$ and $\iiint_D g^2 dV \neq 0$.

Consider the function $f(\alpha) = u\alpha^2 + v\frac{1}{\alpha^2}, \alpha > 0$ where $u > 0, v > 0$, then $\frac{d}{d\alpha}f(\alpha) = 2u\alpha - 2v\frac{1}{\alpha^3}$.

Let $\frac{d}{d\alpha}f(\alpha) = 2u\alpha - 2v\frac{1}{\alpha^3} = 0$, we get when $\alpha^2 = \sqrt{\frac{v}{u}}$ the function f takes its minimal value $2\sqrt{uv}$.

In our case, let $u = \iiint_D f^2 dV$ and $v = \iiint_D g^2 dV$, then $f(\alpha) \geq 2\iiint_D |fg|dV$ implies
$$f((\frac{v}{u})^{\frac{1}{4}}) = 2\sqrt{uv} = 2\sqrt{\iiint_D f^2 dV \iiint_D g^2 dV} \geq 2\iiint_D |fg|dV$$

6.2 Elliptic integral

6.2.1 Perimeter of an eclipse

So how do you understand π, the ratio of a circle's perimeter to its diameter? If you take it for granted, then you could use it to do a lot of things and got many other calculation involving it.

Now think about the perimeter of an eclipse. Do it have the same property as the mysterious number π?

Consider an ellipse with major and minor arcs $2a$ and $2b$ and eccentricity $e := \left(a^2 - b^2\right)/a^2 \in [0, 1)$, e.g.,
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$
What is the arclength $\ell(a; b)$ of the ellipse, as a function of a and b? Let $x = a\cos\theta, y = b\sin\theta, \quad a > b > 0$, then it has length
$$\begin{aligned}\ell(a; b) &= 4\int_0^{\pi/2}\sqrt{dx^2 + dy^2} = 4\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}d\theta \\ &= 4\int_0^{\pi/2}\sqrt{a^2 - (a^2 - b^2)\sin^2\theta}d\theta = 4a\int_0^{\pi/2}\sqrt{1 - e^2\sin^2\theta}d\theta,\end{aligned}$$
Let $z = \sin\theta$, then
$$\begin{aligned}\ell(a, b) &= 4a\int_0^1\sqrt{\frac{1 - ez^2}{1 - z^2}}dz \\ &= 4a\int_0^1\frac{1 - ez^2}{\sqrt{(1 - ez^2)(1 - z^2)}}dz.\end{aligned}$$

We can not find the exact value of $\ell(a,b)$ by hands since the functions $\sqrt{1-e^2\sin^2\theta}$ and $\frac{1-ez^2}{\sqrt{(1-ez^2)(1-z^2)}}$ can be proved to have no elementary anti-derivatives. However, we could assume we know its value just like how we treat π.

6.2.2 Three kinds of elliptic integral

So here is a generalization. An **elliptic integral** is any integral of the general form

$$f(x)=\int\frac{A(x)+B(x)}{C(x)+D(x)\sqrt{S(x)}}dx$$

where $\boldsymbol{A}(\boldsymbol{x}), \boldsymbol{B}(\boldsymbol{x}), \boldsymbol{C}(\boldsymbol{x})$ and $\boldsymbol{D}(\boldsymbol{x})$ are polynomials in $\boldsymbol{x}$ and $\boldsymbol{S}(\boldsymbol{x})$ is a polynomial of degree 3 or 4.

For some cases they are named

Definition 6.1

1. *The **incomplete elliptic integral of the first kind** is defined as*

$$u=F(k,\phi)=\int_0^\phi\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}},\quad 0<k<1,$$

*where ϕ is the amplitude of $F(k,\phi)$ or u, written $\phi=am\ u$, and k is the modulus, $k=\operatorname{mod} u$. The integral is also called Legendre 's form for the elliptic integral of the first kind. If $\phi=\pi/2$, the integral is called the **complete integral of the first kind**, denoted by $K(k)$, or simply K.*

2. *The **incomplete elliptic integral of the second kind** is defined by*

$$E(k,\phi)=\int_0^\phi\sqrt{1-k^2\sin^2\theta}d\theta,\quad 0<k<1,$$

*also called Legendre's form for the elliptic integral of the second kind. If $\phi=\pi/2$, the integral is called the **complete elliptic integral of the second kind**, denoted by $E(k)$, or simply E. This is the form that arises in the determination of the length of arc of an ellipse. For example, $\ell(a,b)=4aE(e)$.*

3. *The **incomplete elliptic integral of the third kind** is defined by*

$$H(k,n,\phi)=\int_0^\phi\frac{d\theta}{\left(1+n\sin^2\theta\right)\sqrt{1-k^2\sin^2\theta}},\quad 0<k<1, n\neq 0,$$

also called Legendre's form for the elliptic integral of the third kind. ♣

Remark If the transformation $v=\sin\theta$ is made in the Legendre forms, we obtain the following integrals, with $x=\sin\phi$

$$F_1(k,x)=\int_0^x\frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$$

$$E_1(k,x)=\int_0^x\sqrt{\frac{1-k^2v^2}{1-v^2}}dv$$

$$H_1(k,n,x)=\int_0^x\frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}$$

called Jacobi's forms for the elliptic integrals of the first, second, and the third kinds respectively. These are complete integrals if $x=1$. The Jacobi's forms conform to the definition of elliptic integral.

In fact, any elliptic integral is a linear combination of elementary functions and the three kinds of elliptic integrals.

By looking at a family of such integration, although we could not obtain the exact value, we can prove

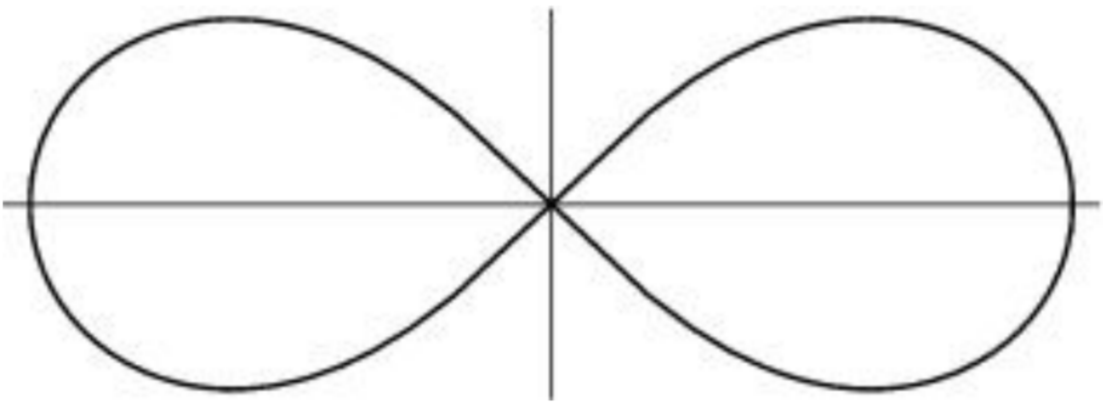

Figure 6.1: Lemniscate

some nice relations between them. The relations are much like the formulas for trigonometric, exponential, or logarithmic functions which also provide many information. We will not discuss these relations but one could refer to http://www.mhtlab.uwaterloo.ca/courses/me755/web_chap3.pdf.

Also one could use the elliptic integral calculator in Mathematica or Matlab to get approximate numerical values.

6.2.3 Arclength of a lemniscate

The lemniscate is the curve: $\left(x^2+y^2\right)^2 = a^2\left(x^2-y^2\right)$, or in polar form $r^2 = a^2\cos 2\theta$. It is the locus of points the product of whose distances from two points (called the foci) is a constant. See figure 6.1

We shall use the arclength formula for polar coordinates:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

Then

$$\begin{cases} \frac{dx}{d\theta} = \frac{\partial x}{\partial r}\frac{dr}{d\theta} + \frac{\partial x}{\partial\theta} = \cos\theta r'(\theta) - \sin\theta r(\theta) \\ \frac{dy}{d\theta} = \frac{\partial y}{\partial r}\frac{dr}{d\theta} + \frac{\partial y}{\partial\theta} = \sin\theta r'(\theta) + \cos\theta r(\theta) \end{cases}$$

So

$$\Rightarrow \sqrt{x'(\theta)^2 + y'(\theta)^2}d\theta = \sqrt{r'(\theta)^2 + r(\theta)^2}d\theta$$

$$\begin{aligned} L = 4\int_{\theta=0}^{\pi/4} ds = 4a\int_{\theta=0}^{\pi/4}\frac{1}{\sqrt{\cos 2\theta}}d\theta = \int_0^{\pi/4}\frac{d\theta}{\sqrt{\cos 2\theta}}, \quad \left(\cos 2\theta = \cos^2 u\right) \Rightarrow \\ = \int_0^{\pi/2}\frac{du}{\sqrt{2-\sin^2 u}} = \frac{1}{\sqrt{2}}\int_0^{\pi/2}\frac{du}{\sqrt{1-\frac{1}{2}\sin^2 u}} = \frac{1}{\sqrt{2}}\cdot K\left(\frac{1}{\sqrt{2}}\right) \end{aligned}$$

Thus, $L = 4a\cdot\frac{1}{\sqrt{2}}K\left(\frac{1}{\sqrt{2}}\right) = a\cdot 2\sqrt{2}(1.85407) = a(5.244102)$.

6.2.4 Arclength of a cubic Bézier curve

Let the curve $(x(t), y(t))$ be defined by polynomials

$$x(t) = a_3t^3 + a_2t^2 + a_1t + a_0$$
$$y(t) = b_3t^3 + b_2t^2 + b_1t + b_0$$

The derivatives are:

$$x'(t) = 3a_3t^2 + 2a_2t + a_1$$
$$y'(t) = 3b_3t^2 + 2b_2t + b_1$$

Squaring these equations gives:

$$\begin{aligned}
\left(x'(t)\right)^2 =& \left(3a_3t^2 + 2a_2t + a_1\right)\left(3a_3t^2 + 2a_2t + a_1\right) \\
=& 9a_3^2t^4 + 6a_3a_2t^3 + 3a_3a_1t^2 + 6a_3a_2t^3 \\
&+ 4a_2^2t^2 + 2a_2a_1t + 3a_3a_1t^2 + 2a_2a_1t + a_1^2 \\
=& 9a_3^2t^4 + 12a_3a_2t^3 + 6a_3a_1t^2 + 4a_2^2t^2 + 4a_2a_1t + a_1^2 \\
\left(y'(t)\right)^2 =& 9b_3^2t^4 + 12b_3b_2t^3 + 6b_3b_1t^2 + 4b_2^2t^2 + 4b_2b_1t + b_1^2
\end{aligned}$$

So the calculation of arclength involves t up to the 4th power. A polynomial of fourth order is used to sort this:

$$L(\tau) = \int_0^\tau \sqrt{c_4t^4 + c_3t^3 + c_2t^2 + c_1t + c_0}dt$$

This is a elliptic integral not of the three kinds. But it can be expressed as a linear combination of elementary functions and the elliptic integral of the three kinds.

6.2.5 Finite-amplitude pendulum.

The equation of motion is:

$$ml\ddot{\theta} = -mg\sin\theta. \text{ Let } p = \dot{\theta} \to p\frac{dp}{d\theta} = -\frac{g}{l}\sin\theta$$
$$\Rightarrow \frac{p^2}{2} = \frac{g}{l}\cos\theta + C.$$

At $t = 0 : \theta = \theta_0, \dot{\theta} = 0 \Rightarrow \frac{d\theta}{dt} = -\sqrt{\frac{2g}{l}}\sqrt{\cos\theta - \cos\theta_0}$. The period, T, is given by

$$\frac{T}{4} = \sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}},$$

or,

$$\begin{aligned}
T &= 4\sqrt{\frac{l}{2g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = 2\sqrt{\frac{l}{g}}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2(\theta_0/2) - \sin^2(\theta/2)}} \\
&= 4\sqrt{\frac{l}{g}}\int_0^{\pi/2}\frac{du}{\sqrt{1 - k^2\sin^2 u}}, \sin\left(\frac{\theta}{2}\right) = \sin\frac{\theta_0}{2}\cdot\sin u, \quad k = \sin\left(\frac{\theta_0}{2}\right)
\end{aligned}$$

$\therefore T = 4\sqrt{\frac{l}{g}}\cdot K(k)$, an elliptic integral. For the special case of small oscillations, $k = 0$, we get the classical result:

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

6.2.6 Pólya's Random Walk Constants

Let $p(d)$ be the probability that a random walk on a d-D lattice returns to the origin. In 1921, Pólya proved that

$$p(1) = p(2) = 1,$$

but

$$p(d) < 1$$

for $d > 2$. Watson (1939), McCrea and Whipple (1940), Domb (1954), and Glasser and Zucker (1977) showed that

$$p(3) = 1 - \frac{1}{u(3)} = 0.3405373296\ldots$$

where

$$\begin{aligned} u(3) &= \frac{3}{(2\pi)^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{dxdydz}{3-\cos x-\cos y-\cos z} \\ &= \frac{12}{\pi^2}(18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})\{K[(2-\sqrt{3})(\sqrt{3}-\sqrt{2})]\}^2 \end{aligned}$$